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The spectral radii of trees

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Abstract Let T be a tree with n vertices and $A(T)$ be the adjacency matrix of T. The spectral radius of $A(T)$, i.e., the largest eigenvalue of $A(T)$, is called the spectral radius of T, denoted by $\rho(T)$. Yu [A. M. Yu, Ordering trees by their spectral radii, *Acta Mathematicae Applicatae Sinica*, 30 (2014) 1107–1112] determined the first ten trees of order n with the smallest spectral radius. In this paper, we extend this ordering by determining the trees with the eleventh to the fifteenth smallest spectral radius among all trees with n vertices.

Keywords Trees; spectral radius; ordering
AMS subject classifications 05C50; 15A18

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices, where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. The maximum vertex degree in G is denoted by $\Delta = \Delta(G)$. For a graph G, let $A(G) = (a_{ij})$ denote the adjacency matrix of G. Since $A(G)$ is a real symmetric matrix, the eigenvalues of $A(G)$ are real numbers. The largest eigenvalue of $A(G)$ is called the spectral radius of G and denoted by $\rho(G)$.

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Let $\mathcal{T}_n$ be the set of trees on n vertices. It is well known that the path P_n is the tree with smallest spectral radius among all the trees in $\mathcal{T}_n$. In [5], Wu et al. determined the first seven trees in $\mathcal{T}_n$ with the smallest spectral radius. In [6], Yu extended this ordering by determining the trees with the eighth to the tenth smallest spectral radius among all the trees in $\mathcal{T}_n$ ($n \geq 21$). In this paper, we prove that $T_{11}(n)$, $T_{12}(n)$, $T_{13}(n)$, $T_{14}(n)$, $T_{15}(n)$ ($T_i(n)$ ($11 \leq i \leq 15$) as shown in Figure 1) are the trees with the eleventh smallest, twelfth smallest, thirteenth smallest, fourteenth smallest, fifteenth smallest spectral radius among all trees of $\mathcal{T}_n$ ($n \geq 25$), respectively.

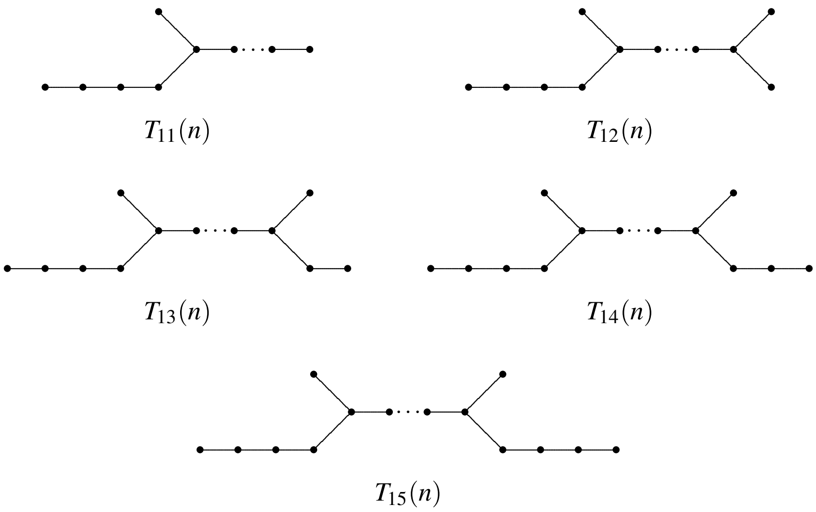

Figure 1

2 Preliminaries

In this section, we give the following lemmas which will be used in the following proof.

Lemma 2.1 [1] *If T' is a proper subgraph of T, then $\rho(T) > \rho(T')$.*

Lemma 2.2 [3] *Denote by $T_{k,\ell}(u)$ the tree obtained from a tree T by attaching paths of length k and ℓ at a vertex u. If $k \geq \ell \geq 1$, then $\rho(T_{k,\ell}(u)) > \rho(T_{k+1,\ell-1}(u))$.*

Let T_n^i ($n \geq 3, 2 \leq i \leq n-1$) be the tree obtained from $K_{1,i}$ by attaching a path of length of $n-i-1$ at one pendant vertex of $K_{1,i}$ (T_n^i as shown in Figure 2). By Lemma 2.2, it is easy to get the following result.

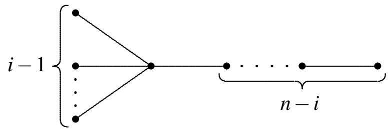

Figure 2. T_n^i

Corollary 2.3 [5] *Let T be a tree in $\mathcal{T}_n$ with maximum degree Δ. Then $\rho(T) \geq \rho(T_n^{\Delta})$, where equality holds if and only if $T \cong T_n^{\Delta}$.*

An internal path of a graph, say $v_1 v_2 \cdots v_{k+1}$ $(k \geq 1)$, is a path joining v_1 and v_{k+1} such that v_1 and v_{k+1} have the degree greater than 2, while all other vertices $v_2, \ldots, v_k$ are of degree 2. A pendant path of a graph is a path where one of its end-vertices has degree one and the other end-vertex has degree greater than or equal to 2, while all the internal vertices have degree two. One can easily see that any edge of T is either on an internal path or on a pendant path of T.

Let $e = uv$ be an edge of T. We denote by $T - uv$ be the subgraph of T obtained by deleting the edge uv.

Lemma 2.4 [2] *Let $e = uv$ be an edge of T and let T' be the tree obtained from T by subdividing uv, i.e., adding a new vertex w and two edges uw, vw to $T - uv$.*

(i) If e is on a pendant path of T, then $\rho(T') > \rho(T)$.

(ii) If e is on an internal path of T and $T \ncong T_3(n)$, then $\rho(T') < \rho(T)$, where $T_3(n)$ is the tree with n vertices as shown in Figure 3.

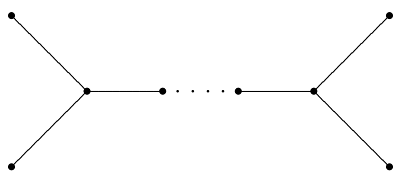

Figure 3. $T_3(n)$

Lemma 2.5 [1] *Let $e = uv$ be an edge of T and let T' be the tree obtained from T by contracting uv, i.e., identifying vertices u and v in $T - uv$.*

(i) If e is on a pendant path of T with length $k \geq 2$, then $\rho(T') < \rho(T)$.

(ii) If e is on an internal path of T with length $k \geq 2$ and $T \ncong T_3(n)$, then $\rho(T') > \rho(T)$.

Corollary 2.6 [6] *Let e_1 and e_2 be two edges of $T \in \mathcal{T}_n$. If e_1 is on a pendant path of T with length $k \geq 2$, e_2 is on an internal path of T, and T' is the tree obtained from T by contracting e_1 and subdividing e_2, then $T' \in \mathcal{T}_n$ and $\rho(T) > \rho(T')$.*

Denote by $T(\ell_1, \ell_2, \ell_3)$ (see Figure 4) the tree with $\ell_1 + \ell_2 + \ell_3 + 1$ vertices consisting of three paths with ℓ_1, ℓ_2 and ℓ_3 edges, respectively, where these paths have one end-vertex in common. Denote by $G(k_1, k_2, k_3)$ (see Figure 4) ($k_1, k_3 \geq 1$, $k_2 \geq 0$) the tree with $k_1 + k_2 + k_3 + 4$ vertices, obtained from a path of $k_1 + k_2 + k_3 + 2$ vertices, by adding pendant vertices at the $(k_1 + 1)$-th and $(k_1 + k_2 + 2)$-th vertex.

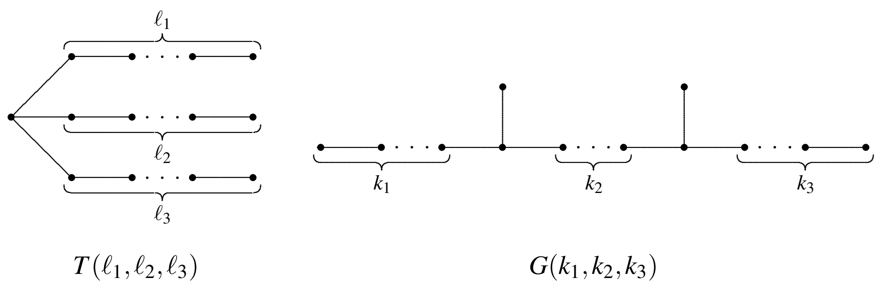

Figure 4

Then we have

$$T_{11}(n) = T(1, 4, n-6), \quad T_{12}(n) = G(4, n-9, 1), \quad T_{13}(n) = G(4, n-10, 2),$$
$$T_{14}(n) = G(4, n-11, 3), \quad T_{15}(n) = G(4, n-12, 4).$$

Lemma 2.7 *If $n \geq 25$, then*

$$\rho(T(1, 4, n-6)) < \rho(G(4, n-9, 1)) < \rho(G(4, n-10, 2))$$
$$< \rho(G(4, n-11, 3)) < \rho(G(4, n-12, 4)).$$

i.e.,

$$\rho(T_{11}(n)) < \rho(T_{12}(n)) < \rho(T_{13}(n)) < \rho(T_{14}(n)) < \rho(T_{15}(n)).$$

Proof. By Lemma 2.2, we have

$$\rho(T(1, 4, n-6)) < \rho(G(4, n-9, 1)).$$

By Lemmas 2.4 and 2.5, we have

$$\rho(G(4, n-9, 1)) < \rho(G(4, n-10, 1)) < \rho(G(4, n-10, 2)).$$

Similarly, we have

$$\rho(G(4,n-10,2)) < \rho(G(4,n-11,2)) < \rho(G(4,n-11,3)).$$
$$\rho(G(4,n-11,3)) < \rho(G(4,n-12,3)) < \rho(G(4,n-12,4)).$$

Hence

$$\rho(T(1,4,n-6)) < \rho(G(4,n-9,1)) < \rho(G(4,n-10,2)) < \rho(G(4,n-11,3)) < \rho(G(4,n-12,4)).$$

i.e.,

$$\rho(T_{11}(n)) < \rho(T_{12}(n)) < \rho(T_{13}(n)) < \rho(T_{14}(n)) < \rho(T_{15}(n)).$$

□

Lemma 2.8 *If $n \geq 25$, then*

(i) $\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.0506$.

(ii) $\rho(T_{15}(n)) < \rho(T_n^4)$, $\rho(T_{15}(n)) < \rho(T(1,5,n-7))$ *and* $\rho(T_{15}(n)) < \rho(T(2,2,n-5))$.

(iii) $\rho(T_{15}(n)) < \rho(T(2,3,3))$.

(iv) $\rho(T_{15}(n)) < \rho(G(1,0,2))$.

Proof. By implementing the newGRAPH program, we have

$$\rho(T_{15}(25)) = 2.05060, \quad \rho(T_{25}^4) \approx 2.12132, \quad \rho(T(1,5,18)) \approx 2.05108,$$
$$\rho(T(2,2,20)) \approx 2.05817, \quad \rho(T(2,3,3)) \approx 2.05288, \quad \rho(G(1,0,2)) \approx 2.05288.$$

By Lemma 2.4, we have $\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060$ and $\rho(T_n^4) \geq \rho(T_{25}^4)$.

So we have

$$\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060 < 2.12132 \approx \rho(T_{25}^4) \leq \rho(T_n^4).$$

Similarly, we have

$$\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060 < 2.05108 \approx \rho(T(1,5,18)) \leq \rho(T(1,5,n-7)),$$
$$\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060 < 2.05817 \approx \rho(T(2,2,20)) \leq \rho(T(2,2,n-5)),$$
$$\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060 < 2.05288 \approx \rho(T(2,3,3)),$$
$$\rho(T_{15}(n)) \leq \rho(T_{15}(25)) = 2.05060 < 2.05288 \approx \rho(G(1,0,2)).$$

□

Lemma 2.9 [4] *If* $n < 2k+3$*,* $1 \leq i < j \leq \lfloor \frac{n-k-4}{2} \rfloor$*, then*

$$\rho(G(i,k,n-4-i-k)) < \rho(G(j,k,n-4-j-k)).$$

Lemma 2.10 *If* $n \geq 25$*,* $k+\ell \geq 8$*,* $G(k,n-k-\ell-4,\ell) \neq G(4,n-12,4)$*, then*

$$\rho(G(k,n-k-\ell-4,\ell)) > \rho(G(4,n-12,4)) = \rho(T_{15}(n)).$$

Proof. Note that $G(4,n-12,4) = T_{15}(n)$. If $k+\ell = 8$, we have

$$\rho(G(k,n-k-\ell-4,\ell)) > \rho(G(4,n-12,4)) = \rho(T_{15}(n))$$

by Lemma 2.9, since $n \geq 25$.

If $k+\ell > 8$, we can obtain $G(k',n-k'-\ell'-4,\ell')$ from $G(k,n-k-\ell-4,\ell)$ by contracting the edges on the pendant paths with length more than 1 and subdividing the edges on the unique internal path such that $k'+\ell' = 8$. By Corollary 2.6, we have $\rho(G(k,n-k-\ell-4,\ell)) > \rho(G(k',n-k'-\ell'-4,\ell'))$. So

$$\begin{aligned}\rho(G(k,n-k-\ell-4,\ell)) &> \rho(G(k',n-k'-\ell'-4,\ell')) > \rho(G(4,n-12,4)) \\ &= \rho(T_{15}(n)).\end{aligned}$$

□

Lemma 2.11 [6] *Let* $T \in \mathcal{T}_n$ *be a tree. If* $n \geq 21$*,* $T \notin \{T_1(n), T_2(n), \ldots, T_{10}(n)\}$*, we have* $\rho(T_1(n)) < \rho(T_2(n)) < \rho(T_3(n)) < \rho(T_4(n)) < \rho(T_5(n)) < \rho(T_6(n)) < \rho(T_7(n)) < \rho(T_8(n)) < \rho(T_9(n)) < \rho(T_{10}(n)) < \rho(T)$*.*

3 Main Results

Lemma 3.1 *Let* $T \in \mathcal{T}_n$*. If* T *has a subgraph isomorphic to* $T(2,3,3)$ *or* $G(1,0,2)$*, then*

$$\rho(T) > \rho(T_{15}(n)).$$

Proof. If T has a subgraph isomorphic to $T(2,3,3)$ or $G(1,0,2)$, by Lemmas 2.1 and 2.8, we have $\rho(T) > \rho(T(2,3,3)) > \rho(T_{15}(n))$ or $\rho(T) > \rho(G(1,0,2)) > \rho(T_{15}(n))$. □

Let $n_3(T)$ be the number of vertices of T with degree 3.

Lemma 3.2 *Let* $T \in \mathcal{T}_n$ *be a tree with* $\Delta(T) = 3$ *and* $n_3(T) = 1$*. If* $n \geq 25$*,* $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$ *and* T *has no subgraph isomorphic to* $T(2,3,3)$ *or* $G(1,0,2)$*, then*

$$\rho(T) > \rho(T_{15}(n)).$$

Proof. Since $\Delta(T) = 3$ and $n_3(T) = 1$, T must be a tree $T(\ell_1, \ell_2, \ell_3)$ as shown in Figure 4, where $\ell_3 \geq \ell_2 \geq \ell_1 \geq 1$ and $\ell_1 + \ell_2 + \ell_3 = n - 1$. Since T has no subgraph isomorphic to $T(2,3,3)$, then $\ell_1 \leq 2$. Now we distinguish the following two cases.

Case 1. $\ell_1 = 1$.

Since $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$, we have $\ell_2 \geq 5$. By Lemmas 2.2 and 2.8, we have $\rho(T) = \rho(T(1, \ell_2, \ell_3)) \geq \rho(T(1,5,n-7)) > \rho(T_{15}(n))$.

Case 2. $\ell_1 = 2$.

Since T has no subgraph isomorphic to $T(2,3,3)$, then $\ell_2 = 2$. By Lemma 2.8, we have $\rho(T) = \rho(T(2,2,n-5)) > \rho(T_{15}(n))$.

The proof is complete. □

Lemma 3.3 *Let $T \in \mathcal{T}_n$ be a tree with $\Delta(T) = 3$ and $n_3(T) = 2$. If $n \geq 25$, $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$ and T has no subgraph isomorphic to $T(2,3,3)$ or $G(1,0,2)$, then*

$$\rho(T) > \rho(T_{15}(n)).$$

Proof. Since $\Delta(T) = 3$ and $n_3(T) = 2$, T must be a tree $H_{\ell_1,\ell_2;\ell_3,\ell_4;k}$ as shown in Figure 5, obtained from a path of $\ell_2 + k + \ell_4 + 2$ vertices by attaching two paths of length ℓ_1 and ℓ_3 to the $(\ell_2 + 1)$-th and $(\ell_2 + k + 2)$-th vertex, respectively, where $\ell_1, \ell_2, \ell_3, \ell_4 \geq 1$ and $k \geq 0$.

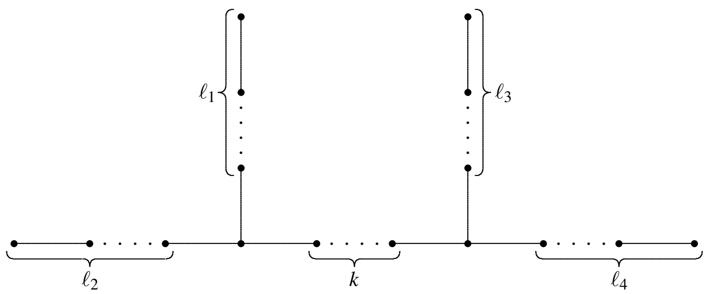

Figure 5. $H_{\ell_1,\ell_2;\ell_3,\ell_4;k}$

Since $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$, there must be some $\ell_i \geq 2$ $(1 \leq i \leq 4)$. Without loss of generality, we assume that $\ell_1 \geq 2$. Since $\ell_1 \geq 2$, and $G(1,0,2)$ is not a subgraph of T, then we have $k \geq 1$. Since $T(2,3,3)$ is not a subgraph of T, then we have $\ell_2 \leq 2$. Now we distinguish the following two cases.

Case 1. $\ell_1 \geq 2$, $\ell_2 = 2$ and $\ell_3 \geq 1$, $\ell_4 \geq 1$.

Noting that $T(2,3,3)$ is not a subgraph of T, we have $\ell_1 = 2$. By Lemmas 2.2 and 2.8, we have $\rho(T) > \rho(T(2, \ell_1, n-3-\ell_1)) = \rho(T(2,2,n-5)) > \rho(T_{15}(n))$.

Case 2. $\ell_1 \geq 2$, $\ell_2 = 1$ and $\ell_3 \geq 1$, $\ell_4 \geq 1$.

Case 2.1. Without loss of generality, we assume that $\ell_3 \geq \ell_4 \geq 2$.

Since $T(2,3,3)$ is not a subgraph of T, we have $\ell_3 = \ell_4 = 2$. By Lemmas 2.2 and 2.8, we have $\rho(T) > \rho(T(\ell_4, \ell_3, n-1-\ell_3-\ell_4)) = \rho(T(2,2,n-5)) > \rho(T_{15}(n))$.

Case 2.2. $\ell_1 \geq 2$, $\ell_2 = 1$ and $\ell_3 \geq 2$, $\ell_4 = 1$.

Without loss of generality, we assume that $\ell_3 \geq \ell_1 \geq 2$. Since $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$, we have $\ell_3 \geq \ell_1 \geq 4$ or $\ell_1 = 2$, $\ell_3 \geq 5$ or $\ell_1 = 3$, $\ell_3 \geq 5$.

Case 2.2.1. $\ell_1 = 2$, $\ell_3 \geq 5$ and $\ell_2 = 1 = \ell_4$.

By Corollary 2.6, Lemmas 2.2 and 2.8, we have $\rho(T) = \rho(G(2, n-6-\ell_3, \ell_3)) \geq \rho(G(2, n-11, 5)) > \rho(T(1,5,n-7)) > \rho(T_{15}(n))$.

Case 2.2.2. $\ell_1 = 3$, $\ell_3 \geq 5$ and $\ell_2 = 1 = \ell_4$.

By Corollary 2.6, Lemmas 2.2 and 2.8, we have $\rho(T) = \rho(G(3, n-7-\ell_3, \ell_3)) \geq \rho(G(3, n-12, 5)) > \rho(T(1,5,n-7)) > \rho(T_{15}(n))$.

Case 2.2.3. $\ell_3 \geq \ell_1 \geq 4$ and $\ell_2 = 1 = \ell_4$.

In this case, $T = G(\ell_1, n-4-\ell_1-\ell_3, \ell_3)$. Since $T \neq G(4, n-12, 4)$, if $\ell_1 = 4$, then $\ell_3 \geq 5$. Noting that $\ell_1 + \ell_3 \geq 9$, by Lemma 2.10, we have $\rho(T) = \rho(G(\ell_1, n-4-\ell_1-\ell_3, \ell_3)) > \rho(T_{15}(n))$.

Case 2.3. $\ell_1 \geq 2$, $\ell_2 = 1$ and $\ell_3 = 1$, $\ell_4 = 1$.

In this case, $T = G(\ell_1, n-5-\ell_1, 1)$. Since $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$, we have $\ell_1 \geq 5$. By Corollary 2.6, Lemmas 2.2 and 2.8, we have $\rho(T) = \rho(G(\ell_1, n-5-\ell_1, 1)) \geq \rho(G(5, n-10, 1)) > \rho(T(1,5,n-7)) > \rho(T_{15}(n))$.

The proof is complete. □

Lemma 3.4 *Let $T \in \mathcal{T}_n$. If $n \geq 25$, $T \notin \{T_1(n), T_2(n), \ldots, T_{15}(n)\}$ and T has no subgraph isomorphic to $T(2,3,3)$ or $G(1,0,2)$, then $\rho(T) > \rho(T_{15}(n))$.*

Proof. Since $T \neq T_1(n)$, we have $\Delta(T) \geq 3$. If $\Delta(T) \geq 4$, by Corollary 2.3, Lemmas 2.2 and 2.8, we have $\rho(T) \geq \rho(T_n^{\Delta}) \geq \rho(T_n^4) > \rho(T_{15}(n))$. If $\Delta(T) = 3$ and $n_3(T) \leq 2$, by Lemmas 3.2 and 3.3, $\rho(T) > \rho(T_{15}(n))$.

If $\Delta(T) = 3$ and $n_3(T) = t \geq 3$, then T must be a tree $P^{m_1, m_2, \ldots, m_t}_{n_1, n_2, \ldots, n_t; p}$ obtained from a path $v_0 v_1 \cdots v_{p-1}$ by attaching one pendant path of length n_i ($n_i \geq 1$) to the vertex v_{m_i}, for each $1 \leq i \leq t$. Otherwise, T has a subgraph T_0 as shown in Figure 6, where $t_1, t_2, t_3 \geq 0$. Since $G(1,0,2)$ is not a subgraph of T, we have $t_1, t_2, t_3 \geq 1$. Then T has a subgraph $T(2,3,3)$. It is a contradiction.

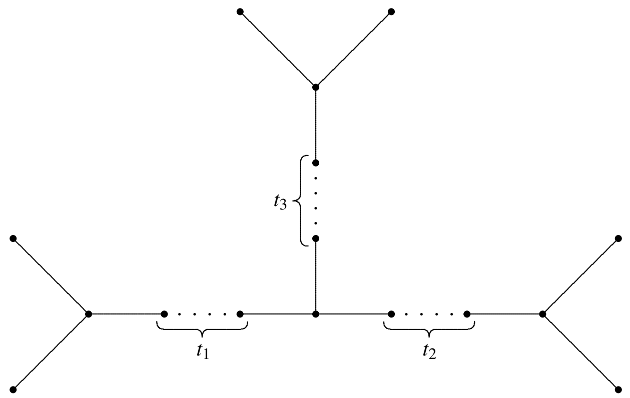

Figure 6. T_0

If there is some $n_i > 1$, using Corollary 2.6 repeatedly, we get a tree $T' = P^{m_1,m'_2,\ldots,m'_t}_{1,1,\ldots,1;p+n_1+\cdots+n_t-t}$ from T such that $\rho(T) > \rho(T')$. So in the following proof, we assume that $T = P^{m_1,m_2,\ldots,m_t}_{1,1,\ldots,1;p}$. For short, we denote $P^{m_1,m_2,\ldots,m_t}_{1,1,\ldots,1;p}$ by $F(k_1,k_2,\ldots,k_{t+1})$ where $k_1 = m_1$, $k_{t+1} = p-(m_t+1)$ and $k_i = m_i-(m_{i-1}+1)$ for $2 \le i \le t$. Since $G(1,0,2)$ is not a subgraph of T, we have $k_i \ge 1$, $1 \le i \le t+1$.

If $n_3(T) = t \ge 4$, using Lemma 2.2 repeatedly, we get a tree $G(k,n-4-k-\ell,\ell)$ from T such that $k,\ell \ge 4$. Since $k+\ell \ge 8$, by Lemma 2.10, we have $\rho(G(k,n-4-k-\ell,\ell)) \ge \rho(T_{15}(n))$. So $\rho(T) > \rho(G(k,n-4-k-\ell,\ell)) \ge \rho(T_{15}(n))$.

If $n_3(T) = t = 3$, then $T = F(k_1,k_2,k_3,k_4)$ as shown in Figure 7. By Lemma 2.2, we have $\rho(F(k_1,k_2,k_3,k_4)) > \rho(G(k_1,k_2,k_3+k_4+2))$. Without loss of generality, we assume that $k_1+k_2 \le k_3+k_4$. Noting that $k_1+k_2+k_3+k_4 = n-6$ and $n \ge 25$, we have $k_3+k_4 \ge 10$ and then $k_1+k_3+k_4+2 \ge 13$. By Lemma 2.10, we have $\rho(G(k_1,k_2,k_3+k_4+2)) > \rho(T_{15}(n))$. So $\rho(T) > \rho(G(k_1,k_2,k_3+k_4+2)) > \rho(T_{15}(n))$. The proof is finished. □

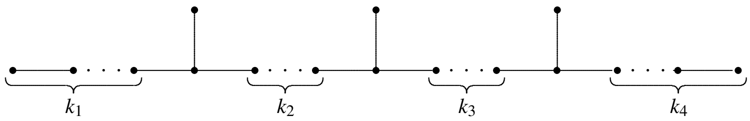

Figure 7. $F(k_1,k_2,k_3,k_4)$

Theorem 3.5 *Let $T \in \mathcal{T}_n$ be a tree. If $n \ge 25$, $T \notin \{T_1(n),T_2(n),\ldots,T_{15}(n)\}$, we have* $\rho(T_1(n)) < \rho(T_2(n)) < \rho(T_3(n)) < \rho(T_4(n)) < \rho(T_5(n)) < \rho(T_6(n)) < \rho(T_7(n)) < \rho(T_8(n)) < \rho(T_9(n)) < \rho(T_{10}(n)) < \rho(T_{11}(n)) < \rho(T_{12}(n)) < \rho(T_{13}(n)) < \rho(T_{14}(n)) < \rho(T_{15}(n)) < \rho(T)$.

Proof. By Lemmas 2.7, 2.11, 3.1–3.4, one can easily see that the result holds. □

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